ABSTRACT

The design and control of underwater robots has to contend with the coupled robot-hydrodynamic interactions. A key aspect of this coupled dynamics is the interaction of the robot with the fluid via the vorticity that is created by the robot’s motion. In this paper we develop a simplified and very low dimensional model of this interaction. This is done recognizing that the vortex shedding is a nonholonomic constraint. We apply the harmonic balance approach to analyze and compare the limit cycle in the dynamics of the fish-shaped body propelled by a periodic input with that of a Chaplygin sleigh, a well known nonholonomic system. The dynamics on the limit cycles lead to a very low dimensional model of the swimming of the fish-shaped body that could be very useful from the perspective of controlling a swimming robot.

1 INTRODUCTION

The design and control of underwater robots have to contend with the coupled robot-hydrodynamic interactions. A key aspect of this coupled dynamics is the interaction of the robot with the fluid via the vorticity that is created by the robot’s motion. It is common in the literature on underwater robots [1–5], to assume that the hydrodynamic forces on a robot consist of a drag and lift force that are assumed to depend on the velocity of the robot and a gross circulation around it. This significant simplification can often ignore aspects of the body-vortex interactions that can be useful for propulsion and maneuvering. In contrast a high fidelity computation of the fluid-body interaction based on the Navier Stokes equation is computationally intensive and challenging. Moreover from the point of view of control and robotics, a simplified and insightful model that captures the essential dynamics of the fluid-body interaction is essential. To this end, the fluid is often assumed to be inviscid with the justification that any viscous effects are important only in a small region around the body. The viscous layer detaches from the body in the vortex shedding process. This process is approximated by the creation of inviscid point vortices at the locations where the vorticity is expected to be created, such as sharp corners on the body. The creation of singular distributions of vorticity is the so called Kutta conditions. This point vortex method has been a useful method to simulate the interaction of a body with vorticity in the fluid [6–12]. In particular the swimming motion of a periodically shape changing Joukowski foil has often been the focus of investigation using the point vortex dynamics, [7, 8, 10–12]. More recently it has been shown through computational methods as well as experiments with a Joukowski foil shaped robot, with a periodic torque applied to the body via the motion of an internal reaction wheel is capable of propelling the body, [11–16].

The point vortex methods are a significant reduction in the modeling of the fluid-body interaction, but still the dimension of the state space of the fluid-body system is very large. Even when the motion of the body is restricted to be planar, the dimension of the state space is equal to $2N + 3$, where is the $N$ number of point vortices. A mathematical model for the body’s motion alone is...
necessary, that nevertheless takes into account the interaction of the body with the vorticity.

In this paper we present such a simplified model, by using the insight that the Kutta condition imposes a nonholonomic constraint on the motion of the body, [13, 17]. The constraint is similar to the constraint on the motion of a well known nonholonomic system, called the Chaplygin sleigh, [18–22]. A modified Chaplygin sleigh, with a quadratic Rayleigh dissipation function was investigated in [23, 24], where it was shown with periodic forcing, the sleigh’s dynamics converge to a limit a cycle in the velocity space. We show through simulations of the Joukowski foil with a periodic torque, the velocity of the foil converges to a limit cycle. We compare the limit cycles of both systems using the harmonic balance method. We show that the dynamics of both systems evolve on a very low finite dimensional attractor, thus allowing for a low dimensional approximation of the Joukowski foil’s motion. We then use this reduced model to control the heading angle of the Joukowski foil.

2 FOIL MODEL AND SIMULATION METHOD

The fish-like robot whose locomotion we are simulating is a non deformable symmetric Joukowski foil. The equations describing the motion of even a simply shaped body through a fluid are difficult to solve due to the complexities of the Navier Stokes equation. Because of this there are numerous numerical approximation methods that can be used to simulate the fluid body interaction. In this paper we utilized a form of the unsteady panel method to run our simulations [25, 26].

In the panel method the body of the foil is broken up into N discrete straight lines (or panels) as shown in Fig. 1. The midpoint of each panel is the control point, which is the point where the zero normal flow boundary condition in the body fixed frame is satisfied. The zero normal flow condition is represented on the ith panel’s control point by,

\[
\sum_{j=1}^{N} A_{i,j}^n (\sigma_j)_k + \gamma_k B_{i,j}^n + \sum_{d=1}^{k-1} C_{i,d}^n L \left( \gamma_d - \gamma_d \right) + W^n L \gamma_{k-1} - \gamma_k = V_i \cdot n_i \tag{1}
\]

In Eqn. 1 \( A^n \) is an N by N matrix where \( A_{i,j}^n \) is the induced normal velocity of a continuous source distribution of unit strength from panel j onto panel i, \( B^n \), \( C^n \) and \( W^n \) are N by 1 vectors describing the induced normal velocity on a control point i from the constant circulation distribution, previously shed point vortices, and current wake panel circulation respectively, \( (\sigma_j)_k \), \( \gamma_k \) represent the current continuous source distribution strength of panel j, and the current circulation around the body. The sum of all the fluid contributions is equal to the \( V \cdot n \) which is the normal velocity of \( i^{th} \) panel. Equation 1 leaves us with N equations and N+1 unknowns, but Eqn. 1 can be rewritten solving for \( \sigma \) as,

\[
\sum_{j=1}^{N} A_{i,j}^n (\sigma_j)_k + \gamma_k B_{i,j}^n + \sum_{d=1}^{k-1} C_{i,d}^n L \left( \gamma_d - \gamma_d \right) + W^n L \gamma_{k-1} - \gamma_k = V_i \cdot n_i \tag{1}
\]

\[
\begin{align*}
\sigma &= (A_{i,j}^n)^{-1} \gamma_k \left( -B_{i,j}^n + W^n L \right) \\
&+ (A_{i,j}^n)^{-1} \left( V \cdot n - \sum_{d=1}^{k-1} C_{i,d}^n L (\gamma_d - \gamma_d) - W^n L \gamma_{k-1} \right) \\
&= b1 \gamma_k + b2.
\end{align*}
\tag{2}
\]

With the source strengths \( \sigma \) written as a function dependent only on the scalar circulation strength \( \gamma_k \), we can now impose the Kutta condition on the trailing edge of the foil. The Kutta condition used in our simulation is that the pressures on the first and last panel must be equal,

\[
||u_1||^2 - ||u_N||^2 = 2 \frac{\partial \Gamma_k}{\partial t} = 2 \frac{(\gamma_k - \gamma_{k-1})}{\Delta t}.
\tag{3}
\]

The Kutta Condition allows us to solve for the circulation strength \( \gamma_k \), for the current time interval, which when plugged back into Eqn. 2 results in the N by 1 vector of source strengths \( \sigma \).

With the circulation strength and source strengths known, the wake panel parameters \( \Delta \), and \( \alpha \) can be determined using

\[
\Delta = \Delta t \sqrt{\frac{V_{xw}^2 + V_{yw}^2}{V_{xw}}}
\]

\[
\alpha = tan^{-1} \left( \frac{V_{yw}}{V_{xw}} \right). \tag{4}
\]
In Eqn. 4 \( V_{xw} \) and \( V_{yw} \) are the fluid velocities at the center of the wake panel in the inertial coordinate frame. The Unsteady Bernoulli equation was then applied around the foil to determine the hydrodynamic forces applied to the body. We assume that the body velocity is constant during each time frame, so we used the backwards finite difference method to determine the bodies new velocity. With updated velocities the code starts over determining the circulation strength, source strength, wake panel parameters and body velocities again until the forces on the body converge. After forces converge, the body and any previously shed point vortices are moved to their new positions, the circulation on the wake panel is concentrated into a point vortex, which is advected into the fluid and the code begins again for the time \( t + \Delta t \). Figure 2 shows a snapshot of the foil immediately after shedding a new vortex into the fluid and progressing to the next time step. In Fig. 2 the blue and red dots represent point vorticies of positive and negative circulation strength respectively.

The rear wheel or runner is not allowed slip in the transverse \((Y_b)\) direction. This yields the nonholonomic constraint

\[
\mathcal{W}(q) \dot{q} = 0, \quad \text{where} \quad \mathcal{W}(q) = [-\sin \theta, \cos \theta, -b] \tag{6}
\]

with Pfaffian one form being

\[
-\sin \theta dx + \cos \theta dy - bd \theta = 0. \tag{8}
\]

The Lagrangian for the system can be written in terms of the state variables as

\[
\mathcal{L} = \frac{1}{2} q^T \mathcal{M}(q) \dot{q}, \quad \text{where} \quad \mathcal{M}(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_c \end{bmatrix}
\]

\[
\mathcal{L}(q) = \frac{1}{2} \dot{q}^T \mathcal{M}(q) \dot{q}, \quad \text{where} \quad \mathcal{M}(q) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_c \end{bmatrix}
\]

A schematic of the Chaplygin sleigh is shown in Fig. 3. The sleigh has a runner or a slender wheel at the rear that contacts the ground at the point \( P \). The runner is assumed to able to slide smoothly in its longitudinal direction but not in a transverse direction. The mass of the sleigh is denoted by \( m \) and the moment of inertia about its center by \( I_c \). A periodic torque \( \tau \) is applied to the body. The configuration of the Chaplygin sleigh is parameterized by the location its the center of mass, \((x,y)\) and its orientation \( \theta \), with respect to an inertial frame of reference. The configuration space of the system is \( Q = SE2 \times S^1 \). The coordinates \((x,y,\theta)\) will be represented by \( q = [q_1,q_2,q_3]^T \) for convenience. The body axes attached to point \( C \) are denoted by \((X_b,Y_b)\).

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\]
The nonholonomic constraint, Eqn. (6) requires the use of the Lagrange multiplier method to derive the equations of motion. Such calculations for the Chaplygin sleigh can be found in, [22, 27] and these can be extended to the case of the Chaplygin sleigh with a passive internal rotor and dissipation. To help us model the dissipative forces the fish experiences by its interaction with the fluid as well as the constraint, we will assume that viscous friction acts on the rear wheel in the longitudinal direction and at the caster on the front link. The viscous damping can be introduced via the Rayleigh dissipation. The Rayleigh dissipation function is

\[ R_w = \frac{1}{2} c u^2 = \frac{1}{2} c(\dot{x} \cos \theta + \dot{y} \sin \theta)^2. \]

where the \( c \) is the damping coefficient. The Euler-Lagrange equations become

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = C_k \dot{\lambda} + Q_{q_k} + \tau_k \]

where \( \lambda \) is the Lagrange multiplier, \( C_k \) is the coefficient corresponding to one forms \( dq_k c \), \( Q_{q_k} = -\frac{\partial \mathcal{L}}{\partial q_k} \) is the dissipation force due to dissipation at the wheel, and \( \tau = [0, 0, \tau]^T \) when \( q_k = \theta \) and \( \tau_k = 0 \) otherwise.

Straightforward forward calculations yield the Euler-Lagrange equations as

\[ \begin{bmatrix} \mathcal{H} - \mathcal{W}^T & 0 \\ \mathcal{W} & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} \dot{\tau} \\ \dot{\varphi} \end{bmatrix} \]

The fifth equation is obtained by differentiating the nonholonomic constraint with respect to time. This is needed to complete the system in this form and solve for the velocities. Before we do so, we eliminate the dependence of the equations on \( \theta \) by writing the equations in terms of \( u \), the velocity of point \( P \) along \( X_x \), and \( \omega = \dot{\theta} \). The velocities and accelerations of the tail may first be expressed in terms of \( u \) and \( \omega \),

\[ \dot{x} = u \cos \theta - \omega b \sin \theta \]

\[ \dot{y} = u \sin \theta + \omega b \cos \theta \]

and

\[ \ddot{x} = \dot{u} \cos \theta - u \omega \sin \theta - \omega^2 b \cos \theta - \omega b \sin \theta \]

\[ \ddot{y} = \dot{u} \sin \theta + u \omega \cos \theta - \omega^2 b \sin \theta + \omega b \cos \theta. \]

After substituting the above expressions into Eqn. (10) and eliminating \( \dot{\lambda} \) the following reduced equations are obtained.

\[ \dot{u} = b \omega^2 - \frac{c}{m} u \]

\[ \dot{\omega} = \frac{-mbu \omega}{I + I_r + mb^2} \]

\[ \dot{\theta} = \omega. \]

In this work we only consider inputs of the form \( \tau = A \cos(\Omega t) \).

A full discussion of stability and dynamics is found in [24]. A description of limiting behavior is found below.

4 HARMONIC BALANCE METHOD FOR SLEIGH

In [23, 24] it was shown that Eqn. (15-17) exhibits stable limit cycles in the \((u, \omega)\) space and that these can be accurately approximated by periodic functions. We briefly review these results. It was shown that the limiting solutions are of the form

\[ u = u_c + A_u \sin(2\Omega t) + B_u \cos(2\Omega t) \]

\[ \omega = A_u \sin(\Omega t) + B_u \cos(\Omega t). \]

To determine \( u_c \) and the coefficients \( A_u, B_u \) we employ the harmonic balance method as follows. Substitute the above equations into Eqn. (15-16). Equating the coefficients of the sine and cosine functions as well as the constant terms yields the following system of nonlinear equations

\[ 0 = A_u^2 Bm + B_u^2 bm - 2cu_c \]

\[ -4m\Omega B_u = 2A_u Bw bm - 2A_u c \]

\[ 4m\Omega A_u = -A_u^2 bm + B_u^2 bm - 2B_u c \]

\[ -2\alpha \Omega B_w = -A_u Bw bm - A_u Bw bm - 2A_u bm uc + 2A \]

\[ 2\alpha \Omega A_w = -A_u A_u bm - B_u Bw bm - 2B_u bm uc. \]

The coefficients can then be found using a nonlinear equations algorithm like the Newton Raphson method. It was also shown that the average velocity of the sleigh on the limit cycle

\[ v_{\text{net}} = \frac{1}{T} \sqrt{(x(t + T) - x(t))^2 + (y(t + T) - y(t))^2}. \]

can be controlled by considering a sixth equation

\[ v_{\text{net}} = \frac{1}{T} \int_0^T u(t) \cos(\theta(t)) dt \]
where \( v_{net} \) is specified and taking \( A \) to be an unknown. Furthermore, the sleigh can be steered by applying an additional torque on the rotor as calculated by

\[
\tau_l = -K_I \int_{t-T}^{t} (\theta(t) - \theta_s) dt
\]  

(23)

where \( \theta_s \) is the desired average heading angle of the sleigh. This control law allows us to control the average velocity and heading of the sleigh simultaneously using \( \tau = A \cos(\Omega t) + \tau_l \). The first term allows us to control the velocity by specifying the amplitude, and the second term allows the system to track a desired average angle.

5 HARMONIC BALANCE FOR FOIL

The results from the panel method simulations, when the input was a periodic torque \( \tau = A \cos(\Omega t) \), showed limit cycle characteristics in the \((u, \omega)\) that were similar to the sleighs. Using the foils simulation results, we estimated the sine and cosine amplitudes \((A_s, B_s, A_w, B_w)\) as well as the velocity constant \( u_c \) from Eqn. (18) and (19). With foil limit cycle coefficients estimated, we applied Eqn. (20), solving for the sleigh parameters \((m, b, c, \alpha)\).

Equations (20) can be rewritten as a set of 5 linear equations if we set \( mb = \delta \). This results in an overdetermined system of 5 equations and only 4 unknowns, whose solutions can be approximated using Matlab’s lsqlin function. The lsqlin function takes a system of linear equations and solves the least-squares problem,

\[
\min_{\vec{x}} \frac{1}{2} ||C \vec{x} - \vec{d}||^2 \quad s.t. \quad A \vec{x} \leq \vec{b}
\]  

(24)

where

\[
C = \begin{bmatrix}
A_s^2 + B_s^2 & 0 & -2u_c & 0 \\
-4A_sB_s & -2B_w & 0 & 0 \\
A_wB_w & 2B_w \Omega & -A_s & 0 \\
A_sA_w + B_sB_w + 2u_cB_w & 0 & 0 & 2A_w \\
-B_sA_w - A_sB_w - 2u_cA_w & 0 & 0 & 2B_w
\end{bmatrix},
\]

\[
\vec{x} = \begin{bmatrix}
\delta \\
m \\
c \\
\alpha
\end{bmatrix}, \quad \vec{d} = \begin{bmatrix}
0 \\
0 \\
0 \\
2A
\end{bmatrix}, \quad A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

The least squares function returned sleigh parameters \( m = 448.28189, \quad b = 0.095293, \quad c = 0.432773, \quad \alpha = 0.00182 \). These parameters were plugged into the sleigh Eqn. (15-16) to get an approximate sleigh model that represents the steady state dynamics of the foil.

The comparison of the limit cycles of the foil and the approximate sleigh are shown in Fig. 4 and 5. In Fig. 4 the blue solid line and red dashed line represent the velocities of the foil and the equivalent sleigh as they approached their limit cycles respectively. It appears that their transient phases are not similar which is to be expected since we are only trying to determine a model of the steady state dynamics. As the sleigh and foil ap-
proach their steady state speeds the figure 8 limit cycles emerge. Fig. 5 shows the limit cycles that the foil in blue and sleigh in red converged to, without the transient velocities present in Fig. 4. It is worth noting that in Fig. 5 the equivalent sleigh doesn’t look like a good match to the foil but this is due to the small resolution of \( u_x \) axis, which is necessary to clearly show the two limit cycles.

Thus far the graphs have only demonstrated qualitatively the similarities between the equivalent sleigh and foil simulation. Because the periodic forcing during the simulations does not allow the foil or the sleigh to travel in a purely straight line, it is beneficial to use their average velocities as a way of comparing their steady state dynamics. Their average velocities or \( V_{net} \) were calculated using Eqn. (22) and the results of their averaged velocities is shown in Fig. 6.

In Fig. 6 the blue dashed line is the average velocity for the foil, while the red solid line is the average velocity of the equivalent sleigh given the same periodic input. It is obvious that the transient dynamics are not related but both the foil and the sleigh’s average velocities converge to nearly the same value, with the difference in the two being \( \approx 0.0006 \text{ [BL/s]} \) or \( \approx 0.18\% \). This proves that the equivalent sleigh model, at least for this specific periodic forcing, does accurately predict the steady state behavior of the foil.

6 TURNING CONTROL

We’ve shown that the sleigh model can accurately predict the average translational velocity of the foil, but can it be used to control the heading angle of the foil also? Equation (23) was used in the equivalent sleigh simulation to drive the sleigh to a heading angle of \( \frac{\pi}{2} \), after it had reached its steady state limit cycle. The same forcing function used in the sleigh simulation was then applied to the foil. The average heading angle from the two simulations is shown in Fig. 7, where the average heading angle (\( \theta_{avg} \)) is the angular position of the body averaged over one time period of the forcing function.

In Fig. 7 the red solid line is the average heading angle for the equivalent sleigh while the blue dashed line is the average heading angle for the foil with the same input. The difference in the final heading angle of the foil and the sleigh was \( \approx 1^\circ \). This shows that the equivalent sleigh model can be used to turn the foil as long as the change in input amplitude only varies slightly, i.e. the foil stays approximately on its limit cycle in the \((u, \omega)\) space.

7 CONCLUSION

We’ve shown that using the equations of motion of a Chaplygin sleigh we can get an approximate dimensionally reduced model describing the steady state dynamics of a Joukowski foil, simulated using the panel method. The approximate model accurately predicts the steady state translational velocity of the foil, and can be used to determine a control input to drive the foils average heading angle to a set reference angle. This will allow for the implementation of real time control of the foil once it has reached its steady state limit cycle.

In future work, the method of model identification in this paper will be expanded upon and refined to develop a model to predict average velocity and heading angle for different periodic forcing functions. After developing a more refined equivalent sleigh model we will apply the same methodology to an experimental robot. Being able to use an approximate sleigh model as a method of feed forward control would be extremely beneficial in a physical aquatic robot where feedback can be limited due to the complexities of flow sensing.
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