The control of the motion of nonholonomic systems is of practical importance from the perspective of robotics. In this paper we consider the dynamics of a cart-like system that is both propelled forward by motion of an internal momentum wheel. This is a modification of the Chaplygin sleigh, a canonical nonholonomic system. For the system considered, the momentum wheel is the sole means of locomotive thrust as well as the only control input. We first derive an analytical expression for the change in the heading angle of the sleigh as a function of its initial velocity and angular velocity. We use this solution to design an open loop control strategy that changes the orientation of the sleigh to any desired angle. The algorithm utilizes periodic impulsive torque inputs via the motion of the momentum wheel.

1 INTRODUCTION

Nonholonomic constraints in mechanical systems are often encountered due to the role that friction plays in rolling motion. Even very simple systems subject to nonholonomic constraints; such as coins and spheres rolling on a rough surface; exhibit very rich and often non intuitive dynamics. From the point of view of robotic platforms involving wheeled motion, it becomes very important to understand these complicated dynamics resulting from nonholonomic constraints. One such possible robotic platform is an actuated version of the Chaplygin sleigh, [1], a canonical example of a mechanical system with nonholonomic constraints.

Zenkov and Osbornne, [2], investigated the motion of a Chaplygin sleigh with a point mass that can be freely moved to anywhere on top of the sleigh and showed that this can be used to steer the sleigh in any direction that one desires. Thus the steering control of the sleigh required two control inputs, the \((x,y)\) coordinates of the point mass. However such a system with a point mass that can be freely moved anywhere on the sleigh is difficult to realize practically. In two recent conference papers, [3, 4], one of the authors and collaborators introduced an approach to steer a Chaplygin sleigh through feedback control. This was done by considering a Chaplygin sleigh with a momentum wheel whose angular acceleration can be controlled. The angular acceleration (or the torque on the momentum wheel) is the single control input and the heading angle of the sleigh is the output. For this system it was shown that if the angular acceleration of the momentum wheel was proportional to the error in the heading angle of the sleigh, then one could steer the sleigh to any desired direction. However the drawback of the proportional controller is that the momentum wheel continues to rotate with a constant angular velocity even after the desired heading angle was reached. This continuous actuation of the momentum wheel to maintain a constant heading angle is very inefficient.

The intrinsic dynamics of the Chaplygin sleigh can be useful in steering its motion in a desired direction. We investigate these intrinsic dynamics by deriving an analytical expression for the change in the heading angle of the sleigh as a function of the sleigh’s initial velocity and angular velocity. Such an analysis can be found in [5, 6] which we adapt to the modified Chaplygin sleigh with a momentum wheel.
wheel. We then propose an open loop method to steer the Chaplygin sleigh by the application of a sequence of periodic impulses that change the angular velocity of the sleigh. This is a simplified variation of the well known method of steering wheeled nonholonomic systems through sinusoidal control inputs to the wheels, [7, 8]. The efficacy of the proposed method is demonstrated with three examples of maneuvers; a change in heading angle of the sleigh by 60°, an about turn by the sleigh and a parallel displacement. The advantage of the method proposed is that the control input is zero most of the time and remains zero once when the heading angle of the sleigh reaches a neighborhood of the desired angle.

2 Background on the Chaplygin sleigh

The Chaplygin sleigh consists of a cart with a knife edge (or a wheel) in contact with the ground, as shown in Fig. 1. The coordinates of the center of the sleigh are denoted by (x,y) in an inertial frame. The body frame is denoted by the axes Xb and Yb centered at (x,y). The point of contact of the wheel is denoted by P. The wheel is assumed to never slip in the transverse direction (along Yb) but is free to roll without slipping in the longitudinal direction (along Xb). At the front of the sleigh are castors that allow rotation in any direction. This classical model is modified with the addition of a balanced rotor of moment of inertia IR, placed at the center of mass of the sleigh. The distance of the center of mass of the combined system from the wheel is denoted by b, the position of the center of the sleigh by (x(t),y(t)) and its orientation by θ(t).

Fig. 1. Chaplygin sleigh with a balanced rotor. The rotor is placed at distance of b from the rear contact. The centers of mass of both sleigh and the momentum wheel coincide at (x,y). The no-slip constraint is enforced at point P. The picture on the right shows an illustration of a physical cart to realize the Chaplygin sleigh with castors at the front.

The constraint of no-slip in the transverse direction at point P can be written as

\[-\dot{x}\sin\theta + \dot{y}\cos\theta - b\dot{\theta} = 0.\]  (1)

The physics of the system is as follows. If the rotor is forced to rotate with an angular velocity of \( \dot{\phi} \) relative to the cart, then due to the conservation of angular momentum the sleigh would counter rotate in the absence of friction. However in the presence the no slip constraint at the point P, the sleigh is forced to translate in a manner so as to cancel the velocity at point P. It is important to note that in the absence of the no-slip constraint at point P, the angular motion of the rotor would merely lead to the angular motion of the cart about its center.

The following derivation of the equations of motion of the cart are follows [9] in which a similar system was investigated. Alternative derivations of the reduced momentum equations, [10], can be found in [4]. The Lagrangian for the system is

\[L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_c \dot{\theta}^2 + I_R(\dot{\theta} + \dot{\phi})^2\]

where \( I_c \) is the moment of inertia of the cart alone and \( m \) is the combined mass of the cart and the rotor. The Euler Lagrange equations are

\[m\ddot{x} = -\lambda \sin\theta \]  (2)
\[m\ddot{y} = \lambda \cos\theta \]  (3)
\[(I_c + IR)\ddot{\theta} + I_R\ddot{\phi} = -b\lambda. \]  (4)

where \( \lambda \) is a Lagrange multiplier. We will denote the combined moment of inertia of the sleigh and rotor about the center of mass by \( I = I_c + IR \).

Denoting the velocity of the sleigh at the point of contact, P, by \( (u_x, u_y) \) in the body frame, the velocities of the center of the cart can be expressed as

\[\dot{x} = u_x \cos\theta - \omega b \sin\theta \]  (5)
\[\dot{y} = u_x \sin\theta + \omega b \cos\theta \]  (6)
\[\omega = \dot{\theta}. \]  (7)

These equations can be differentiated with respect to time in order to obtain \( \ddot{x} \) and \( \ddot{y} \),

\[\ddot{x} = \dot{u}_x \cos\theta - u_x \omega \sin\theta - \omega^2 b \cos\theta - \omega \dot{b} \sin\theta \]  (8)
\[\ddot{y} = \dot{u}_x \sin\theta + u_x \omega \cos\theta - \omega^2 b \sin\theta + \omega \dot{b} \cos\theta. \]  (9)

Multiplying (2) by \( \cos\theta \) and (3) by \( \sin\theta \) and adding both gives

\[\dot{u}_x = b\omega^2. \]  (10)

Multiplying (2) by \( \sin\theta \) and (3) by \( \cos\theta \) and adding both gives \( \dot{\lambda} = m(\dot{y}\cos\theta - \ddot{x}\sin\theta) \). Substituting this into (4) gives

\[\frac{d\omega}{dt} = -\frac{mb}{mb^2 + I} \dot{u}_x + \frac{T}{mb^2 + I} \]  (11)

where \( T = -I_R\ddot{\phi} \) is the torque that the motion of the momentum wheel exerts on the sleigh. This torque is directly proportional to the angular acceleration of the momentum wheel. Thus one can imagine exerting a torque on the sleigh through the actuation of the momentum wheel. The motion of the sleigh in the plane can be obtained from the (5), (6) and (7).
3 Change in the heading angle of the Chaplygin sleigh

In the case of zero torque, several interesting observations can be made. The first observation is that the \( u_x \) is a monotonically increasing function of time since the right hand side of (10) is nonnegative. The fixed points of (10) and (11) lie on the line \( \omega = 0 \); i.e for any initial conditions \((u_x(0), \omega(0))\) the motion of the sleigh asymptotically approaches a straight line. In subsequent calculations the kinetic energy of the sleigh will be shown to be a constant of motion. This means that any initial angular velocity of the sleigh decays to zero and the speed of the cart, \( u_x \) increases in a manner such that the total kinetic energy is conserved. Therefore the asymptotic value of \( u_x \) is easily known.

Making use of these observations, we show that the orientation angle of the sleigh \( \theta(t) \) can be exactly calculated from which the asymptotic change in the heading angle can be obtained. It should be mentioned that our calculation of \( \theta(t) \) is inspired by the one in [6] which however suffers from an unfortunate error, that implied that the change in heading angle of the cart is independent of its initial angular velocity.

In the case of zero torque, \( T = 0 \), equations (10) and (11) can be simplified as follows. Rearranging the terms in (11) and multiplying by \( \omega \), we obtain,

\[
b K^2 \frac{d}{dt} \left( \frac{\omega}{\omega} \right) + \frac{du_x}{dt} = 0
\]

where \( K^2 = 1 + \frac{I}{mb^2} \). Substituting for \( \omega \) from (10) into (12) and multiplying by \( \frac{\omega}{\omega} \), one obtains,

\[
K^2 \left( \frac{\omega}{\omega} \right) \frac{d}{dt} \left( \frac{\omega}{\omega} \right) = -\omega \frac{d\omega}{dt}.
\]

Integrating (13) gives

\[
K^2 \left( \frac{\omega}{\omega} \right)^2 = K^2 C^2 - \omega^2
\]

where \( C \) is a constant of integration. Observing that \( \frac{\omega}{\omega} = \frac{\omega}{K}$, it can be verified that

\[
C^2 = \frac{1}{K^2} \left( \omega^2 + \frac{u_x^2}{b^2 K^2} \right) = \frac{1}{K^4 mb^2 \pi} (I + mb^2) \omega^2 + mu_x^2
\]

showing that the constant \( C \) is merely the kinetic energy scaled by the factor \( \frac{2}{K^3 mb^2 \pi} \). The equation (14) can be simplified by setting \( \omega(t) = KC \cos \phi(t) \) to give

\[
\phi = \pm C \cos \phi
\]

which can be integrated to give

\[
\sin \phi = \frac{e^{2Ct} - A}{e^{2Ct} + A}
\]

where the constant \( A \) is defined as

\[
A = \frac{1 - \sin \phi(t_0)}{1 + \sin \phi(t_0)}
\]

and \( \phi(t_0) = \cos^{-1} \left( \frac{\omega(t_0)}{KC} \right) \). Now one can finally derive an expression for \( \theta(t) \) using (16) as follows;

\[
\omega = \frac{d\theta}{dt} = KC \cos \phi = \pm K \frac{d\phi}{dt}
\]

integrating which gives,

\[
\Delta \theta = \theta(t) - \theta(t_0) = \pm K \left( \frac{\pi}{2} - \phi(t_0) \right)
\]

The limiting value of the change in the orientation angle is,

\[
\Delta \theta = \lim_{t \to \infty} (\phi(t) - \phi(t_0)) = \pm K \left( \frac{\pi}{2} - \phi(t_0) \right)
\]

The plus sign on the left hand side of (21) is chosen when \( \omega(0) \) is positive and the negative sign is chosen when \( \omega(0) \) is negative. The value of \( \Delta \theta \) is bounded above by \( K \frac{\pi}{2} \). This occurs when \( u_x(t_0) = 0 \); i.e. when \( \omega(t_0) = KC \).

Equations and (20) and (21) are exact analytical expressions for the orientation angle of the sleigh and its corresponding asymptotic orientation. These expressions provide a means for the open loop control of the heading angle of the sleigh. For example the change in angle \( \Delta \theta \) is plotted as a function of \( u_x(0) \) and \( \omega(0) \) in Fig. 2 for \( K = \sqrt{2} \).

Fig. 2. The asymptotic value of \( \Delta \theta \) as a function of \( u_x(0) \) and \( \omega(0) \); obtained from equation (21). The data corresponds to \( K = \sqrt{2} \).
4 Steering the Chaplygin sleigh through impulsive torques

We consider the problem of steering the Chaplygin sleigh with impulsive torques applied through the momentum wheel. This simplification models the scenario where the momentum wheel is in motion for a brief period of time, during which the angular momentum (and thus the angular velocity) of the Chaplygin sleigh is modified very quickly. We model these rapid bursts of motion of the momentum wheel via generalized Dirac δ-functions. The admissible control inputs to the system are of the form

\[ T(t) = \sum_{n=0}^{N} T_n \delta(t - n\tau) \]  

(22)

where \( T_0 \in [0, T_{\text{max}}] \), \( N \) is a positive integer, \( \delta(t - n\tau) \) is the generalized Dirac delta function and \( \tau \) is a known parameter that defines the momentum wheel’s minimum relaxation time between two consecutive torque impulses.

Suppose the initial conditions of the system described by equations (10) - (6) are \((u_x(0), \omega(0) = 0, \theta(0) = 0, x(0), y(0))\). We seek to construct an admissible control torque input function such that

\[ \lim_{t \to \infty} \theta(t) = \theta_r. \]  

(23)

We note that the goal is restricted to controlling the orientation of the sleigh and not its speed or its path.

It is shown in [2] and [4], that the angular momentum of the sleigh about the point of contact \( P \) is

\[ p_\theta = (I + mb^2)\omega. \]  

(24)

The effect of an impulsive torque \( T\delta(t) \) is to instantaneously change the angular momentum of the sleigh by an amount, \( \Delta p_\theta = T \). Therefore the angular velocity of the sleigh changes instantaneously by an amount \( \Delta \omega = \frac{T}{I + mb^2}. \) The same is implied by (11), where the torque is an impulsive function \( T\delta(t) \). Therefore the problem of finding a sequence of impulsive torques is equivalent to finding a set of admissible perturbations to the angular velocity, \( \Delta \omega \), of the sleigh of the form

\[ \Delta \omega(t) = \sum_{n=0}^{N} \Delta \omega_n \delta(t - n\tau) \]  

(25)

with \( \Delta \omega_n \in [0, \Delta \omega_{\text{max}}] \) where \( \Delta \omega_{\text{max}} = \frac{T_{\text{max}}}{I + mb^2}. \)

Suppose at time \( t_0 \), the velocity of the point \( P \) on the sleigh is \( u_x(0) \) and the angular velocity of the sleigh is \( \omega(t_0) \).

In a time interval of \( \tau \), the relaxation time of the actuator, the sleigh rotates by an angle denoted by \( \Delta \theta(\tau) \) given by,

\[ \Delta \theta(\tau) = K(\phi - \phi_0) \]  

(26)

where

\[ \phi = \sin^{-1} \left( \frac{e^{2C\tau - A}}{e^{2C\tau + A}} \right) \]

with \( A = \frac{1 - \sin \phi_0}{1 + \sin \phi_0} \) and \( \phi_0 = \cos^{-1} \left( \frac{\omega(t_0)}{KC} \right) \). In the case where the angular velocity of the sleigh is perturbed at regular intervals of time, the angle by which the sleigh rotates between consecutive perturbations can be exactly know through (26).

Correspondingly the perturbation in angular velocity at \( t_0 \) that produces a specified asymptotic rotation \( \Delta \theta \) can be calculated from (21) and (15),

\[ \Delta \omega = \frac{u_x(t_0)}{bK} \tan \left( \frac{\Delta \theta}{bK} \right) - \omega(t_0) \]  

(27)

Knowing the formulas (26) and (27), an algorithm utilizing periodic impulsive perturbations to the angular velocity is proposed.

The inputs to the algorithm are the initial kinematic variables of the sleigh, \((u_x(0), \omega(0), \theta(0))\), the desired heading angle \( \theta_r \), the maximum allowable perturbation to the angular velocity, \( \Delta \omega_{\text{max}} \) and the relaxation time \( \tau \). The error in the angle is \( \theta_e = \theta_r - \theta(t) \). If this error in the angle can be eliminated with one perturbation of the angular velocity, \( \Delta \omega \) such that the perturbation does not exceed \( \Delta \omega_{\text{max}} \), the perturbation is applied and the sleigh asymptotically steers to the desired heading angle, \( \theta_r \). The condition is checked using the formulas in (26) and (27). If the required perturbation in the angular velocity is more than the maximum allowed, then the maximum perturbation, \( \Delta \omega_{\text{max}} \) is applied. The change in the heading angle of the sleigh due to this perturbation in a time interval of \( \tau \) is calculated to obtain the new error angle. The process is repeated until the first condition \( \Delta \omega \leq \Delta \omega_{\text{max}} \) is met and a perturbation of \( \Delta \omega \leftarrow \frac{\theta_e}{bK} \tan \left( \frac{\theta_e}{bK} \right) - \omega \) is applied.

5 Simulation

We demonstrate the use of our control algorithm with three examples of steering the Chaplygin sleigh. In all the cases the relaxation time \( \tau = 0.2 \) and \( \Delta \omega_{\text{max}} = 0.1 \).

In the first example, \( u_x(0) = 1, \omega(0) = 0, \theta(0) = 0 \) and \( \theta_e = \frac{\pi}{4} \) for a sleigh with the parameter \( K = \sqrt{2} \). The algorithm generates 7 impulsive inputs to steer the sleigh in the desired direction. The kinematic variables \( \omega(t), \theta(t) \) and path of the sleigh are shown in Fig. 5. As observed earlier, the trajectory of the reduced system (10) and (11) converge to the set \( \omega = 0 \).

In the second example, \( u_x(0) = 5, \omega(0) = 0, \theta(0) = 0 \) and \( \theta_e = \pi \), that is sleigh is required execute an about turn. The initial velocity of the sleigh is chosen to be higher than in the previous example. This necessitates a higher number of impulsive inputs, since the change in heading angle produced by each perturbation to the angular velocity decreases as the velocity \( u_x \) increases. The kinematic variables for this case are shown in Fig. 5.
In the third example we consider the case where the sleigh is required to achieve a net displacement parallel to itself. This is a variation on the theme of the problem parallel parking a car and demonstrates the possibility of the algorithm to steer the cart clear of obstacles. The initial conditions are $u_x(0) = 1.0$, $\omega(0) = 0$ and $\theta(0) = 0$. The parallel motion maneuver is executed by first steering the sleigh by an angle of $\frac{\pi}{6}$ in the counterclockwise direction and then steering back by an angle of $\frac{\pi}{6}$ in the clockwise direction. The choice of the angle $\frac{\pi}{6}$ is merely to demonstrate a noticeable parallel translation.

6 Conclusion

In this paper we revisited some of the calculations associated with canonical problem of the motion of a Chaplygin sleigh. We derived an exact expression to find out the change in the heading angle of a Chaplygin sleigh for any initial conditions. We then used this solution to develop an open loop control algorithm to steer a Chaplygin sleigh to any desired heading angle. The control inputs are impulsive torques applied via the impulsive motion of a balanced momentum wheel. These impulsive torques acting at regular intervals of time produce perturbations in the angular velocity of sleigh such that it is steered towards the desired angle. The impulsive perturbations are such that once the sleigh reaches the desired heading angle, no more control inputs are necessary to maintain a heading angle. The results presented in this paper are expected to be useful in developing strategies for the control of the motion of robotic systems inspired by the Chaplygin sleigh.

References

